The robustness of democratic consensus

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Abstract

In linear models of consensus dynamics, the state of the various agents converges to a value which is a convex combination of the agents' initial states. We call it democratic if in the large scale limit (number of agents going to infinity) the vector of convex weights converges to 0 uniformly.

Democracy is a relevant property which naturally shows up when we deal with opinion dynamic models and cooperative algorithms such as consensus over a network: it says that each agent's measure/opinion is going to play a negligeable role in the asymptotic behavior of the global system. It can be seen as a relaxation of average consensus, where all agents have exactly the same weight in the final value, which becomes negligible for a large number of agents.

We prove that starting from consensus models described by time-reversible stochastic matrices, under some mild technical assumptions, democracy is preserved when we perturb the linear dynamics in finitely many vertices. We want to stress that the local perturbation in general breaks the time-reversibility of the stochastic matrices. The main technical assumption needed in our result is the irreducibility of the large scale limit stochastic matrix , i.e. strong connectedness of the limit network of agents, and we show with an example that this assumption is indeed required.

Key words: consensus; Markov chain; perturbation

1 Introduction

1.1 Consensus

Many opinion dynamics models [11], [13] and cooperative algorithms over networks like consensus [14], [18], [5] are mathematically represented by a stochastic matrix $P \in \mathbb{R}^{V \times V}$ where V is a finite set. Interpreting x_i as an initial belief/opinion of agent $i \in V$ on some fact or event, or a position in a physical space, linear consensus dynamics consists in replacing each opinion x_i by a weighted average of the opinion of agent i's neighbors in the network. Such dynamics may be expressed by the equation $x(t+1) = Px(t)$ where P is row-stochastic, i.e., is nonnegative with every row summing to one. Another motivation is the design or analysis of agents such as robots moving on the real line or any Euclidean space, while exchanging messages on their respective positions in on a communication network. The equation $x(t+1) = Px(t)$ now describes the situation where every agent moves to a weighted average of the position of their neighbors in the network. The robots typically seek to solve the consensus problem, i.e. to all reach a common position in the space.

It is well known that under suitable assumptions on P (i.e. irreducibility and aperiodicity) there exists $\pi \in \mathbb{R}^V$ such that

$$\lim_{t\to+\infty} (P^t x)_i \to \sum_{j\in V} \pi_j x_j(0)\,, \quad \forall i \in V\,. \qquad (1)$$

Moreover, $\pi_i > 0$ for all $i \in V$, $\sum_i \pi_i = 1$ and $\pi^* P = \pi^*$, where π^* denotes the transpose of π and is thus a row vector.

In terms of consensus or opinion dynamics, convergence (1) means that the opinion of all agents tends to the common value $\sum_{j\in V} \pi_j x_j$ which is a convex combination of the initial opinions. For this reason, in this paper,

a stochastic matrix P for which (1) holds will be called a *consensus matrix* and the relative vector π the corresponding *consensus weight vector* of P. If π is the uniform vector (i.e. $\pi_i = |V|^{-1}$ for all i), the common asymptotic value is simply the arithmetic mean of the initial beliefs; in other terms, all agents equally contribute to the final common belief. This uniformity condition amounts to assuming that the matrix P is doubly stochastic (also all columns sum to 1), a sufficient condition for this being that P is symmetric.

In this paper we want to consider the situation where we have a sequence $P^{(n)}$ of consensus matrices over a state space V_n of increasing cardinality corresponding to larger and larger sets of interacting agents. The corresponding consensus weight vectors will be denoted by $\pi^{(n)}$.

1.2 Democracy

The sequence $P^{(n)}$ of consensus matrices is called *democratic* if their corresponding invariant probabilities $\pi^{(n)}$ are such that $||\pi^{(n)}||_\infty := \max_{i\in V_n} \pi_i^{(n)} \to 0$ for $n \to +\infty$. This says that even if the initial opinion of the various agents may have a different weight on the final consensus value, still the weight of each of them becomes negligeable as the number n of agents grows to ∞. This property has already been proposed in [11], [13] as 'wise society' with the following interpretation. If we assume that the initial opinion of the various agents are of type $x_i = \mu + N_i$, where $\mu \in \mathbb{R}$ is the value of a parameter we want to estimate and N_i are independent noises having mean 0 and variance σ_i^2, then, the consensus point reached by applying the consensus matrix $P^{(n)}$ is given by

$$\sum_j \pi_j^{(n)} x_j = \mu + N\,, \quad \text{with } N = \sum_j \pi_j^{(n)} N_j$$

If σ_i^2 are bounded from above, it follows from a straightforward variation of the weak law of large numbers [11] that democracy implies that $N \to 0$ in probability when $n \to +\infty$. In wise societies agents' asymptotic belief converge to the real value of the parameter when the number of agents goes to ∞.

A very special case is when we start from a sequence $G^{(n)}$ of connected undirected graphs (with no self loops) on the set of vertices V_n and the consensus matrices $P^{(n)}$ are obtained by assigning homogeneous weights to all neighbors of an agent. Put $d_i^{(n)}$ equal to the degree in $G^{(n)}$ of the vertex i (number of edges connected to i) and define

$$P_{ij}^{(n)} = \frac{1-\tau}{d_i^{(n)}} \quad \text{for } j \text{ neighbor of } i, \quad P_{ii}^{(n)} = \tau \qquad (2)$$

while $P_{ij}^{(n)} = 0$ if $j \neq i$ is not a neighbor of i in $G^{(n)}$, where $0 \le \tau < 1$ is a self-confidence parameter (see e.g. [9,10] for other models of self-confidence, or stubborness, in opinion dynamics). In this case we have that $\pi_i^{(n)} = d_i^{(n)} / \sum_j d_j^{(n)}$. In this context, democracy thus happens to be a rather easily checkable property only depending on the degrees of the various nodes. In particular, if graphs are regular ($d_i^{(n)}$ constant in i) the consensus weight vectors all coincide with the uniform one. More generally, if we have a uniform bound $d_i^{(n)} \le d$ for all n and $i \in V_n$, then, clearly, $||\pi^{(n)}||_\infty$ goes to 0. This example is encompassed by the more general time-reversible consensus matrices which will be revised in next section. For them, an explicit characterization of the consensus weight vectors remains available so that $||\pi^{(n)}||_\infty$ can be estimated and democracy can easily be checked. Quite a different story is when time-reversibility is lost (e.g. sequences $P^{(n)}$ constructed as in (2) over directed graphs $G^{(n)}$): in this case there is no general techniques available to characterize the consensus weights vectors and check democracy.

In [11] the authors propose a sufficient condition for democracy (see their Theorem 1) which can be applied also to stochastic matrices which are not time-reversible. However, one of their assumptions (Property 2) never holds when the underlying sequence of graphs have a bounded degree and this rules out many interesting examples.

1.3 Robust democracy and main result

In this paper we focus on the robustness of democracy with respect to local perturbations. More precisely, we start from a democratic sequence $P^{(n)}$ defined on a sequence of nested sets V_n of nodes (i.e., $V_n \subset V_{n+1}$) and we analyze what happens to the consensus weights vectors when $P^{(n)}$ is locally perturbed. The perturbed sequence of consensus matrices $\tilde{P}^{(n)}$ coincides with $P^{(n)}$ but in a fixed finite number of rows corresponding to a subset of vertices W.

Our Theorem 2 shows that under very mild assumptions (irreducibility of the limit chains , i.e. strong connectedness of the limit graph) $\tilde{P}^{(n)}$ maintains a weak form of democracy (pointwise convergence to 0 of the consensus weight vectors). Afterwards, we focus on time-reversible chains $P^{(n)}$ and in Theorem 3 we prove that, under some technical assumptions (essentially that degrees are bounded in the associated graphs) the perturbed sequence $\tilde{P}^{(n)}$ (possibly no longer time-reversible) remains democratic. We again want to stress the fact that the sufficient conditions for democracy proposed in [11] can not be applied in this context as their property 2 will never be satisfied. The proofs of these results will be probabilistic in nature interpreting $P^{(n)}$ and $\tilde{P}^{(n)}$ as transition matrices of Markov chains and the corresponding

consensus weights as invariant probability vectors. Although our motivation and applications for our results lie in the field of opinion dynamics and consensus, we find the dual language of Markov chains more convenient and powerful to express the technical results and proofs.

1.4 Applications and context

From the point of view of opinion dynamics, these results essentially say that in democratic chains, no single agent or a finite group of them can unilaterally break democracy by modifying their outgoing links or weights as long as the number of links remains bounded and the graph connected.

As a more specific example, we can consider a sequence of connected undirected graphs over a nested set of vertices V_n and $P^{(n)}$ to be the corresponding consensus matrices as defined in (2). Fix now a subset $W \subseteq V_1$ and perturb $P^{(n)}$ on W by assuming that agents in W form a small community which is incline d to give more credit to each other' s opinion than to people outside of W. This can be modeled by simply assuming that, for each $i \in W$, all weights $\tilde{P}^{(n)}_{ij}$ for $j \in W$ are a factor $\lambda \geq 1$ greater than weights $\tilde{P}^{(n)}_{ij}$ for $j \notin W$. The parameter λ, called 'homophily', measures the 'closure' of the community W to external influence. Our results assert that, disregarding how large λ is, democracy is preserved: in the final consensus the opinion of these agents still plays a negligeable role when $|V_n| \to +\infty$. This example is treated in a more formal way in Section 2 (see Example 5).

Related perturbation problems in the context of opinion dynamics have been considered in [1] where the authors study a novel gossip consensus model where a limited number of pairwise interactions are asymmetric (one of the two agents engaged in the interaction, called forceful, does not change opinion). The mean behavior of agents is governed by a stochastic matrix $\tilde{P}$ which can be represented as the perturbation of a symmetric one P (corresponding to the situation where all interactions are symmetric). Clearly, the consensus weight vector of P is the uniform one $\pi_i = N^{-1}$ where N is the number of nodes. Their main results (Theorems 5 and 6 therein) are explicit bounds of the distance between $\tilde{\pi}$ and π in the infinity and in the 2 norm. Connection with democracy can be obtained through the following straightforward inequalities

$$||\tilde{\pi}||_\infty - \frac{1}{N} \leq ||\tilde{\pi} - \pi||_\infty \leq ||\tilde{\pi}||_\infty + \frac{1}{N}$$

which implies that democracy can be equivalently expressed, in this context, by $||\tilde{\pi} - \pi||_\infty \to 0$ (in correspondence of larger and larger graphs). Similar considerations also apply to the 2-norm. Our results allow to conclude that when the set of forceful agents remains fixed and finite, the perturbed mean behavior remains democratic (see Theorem 3 in this paper). This implies, in particular, that the perturbed invariant probability $\tilde{\pi}$ converges to the uniform one in the infinity norm. This convergence cannot be deduced in general directly from their Theorems 5 or 6 as their estimation contains a critical parameter at the denominator (the spectral gap in Theorem 5) which may be infinitesimal for certain families of graphs like grids. On the other hand, it is important also to remark that our result is only asymptotical and, differently from theirs, it does not lead to any explicit bound on the distance between consensus vectors.

See also [2] for related results on the analysis of gossip consensus algorithms in the presence of stubborn agents who never modify their opinion.

The type of perturbations discussed in this paper have also a considerable importance in other contexts. When a consensus algorithm is implemented into a real physical network of communicating robots, it is possible that certain communications are down in one direction, or that, in any case, messages are lost in one direction. Even if the underlying stochastic matrix was designed to be reversible, it is therefore possible that the actual algorithm will follow the dynamics of a perturbed stochastic matrix which is no longer reversible. In this application it is important to avoid the situation where all agents converge to one single immobilized agent, resulting in a possible waste of energy. This case is easily ruled out by restricting our attention to perturbations that leave the network strongly connected, or the corresponding matrix irreducible, as it prevents any node from being stripped of all incoming edges. It is desirable to find supplementary conditions ensuring that if a small number of communication channels break down, the final consensus position is not significantly far away from the arithmetic average of the agents' initial position, which is typically optimal in terms of resources.

Another application regards the webmaster problem [6,12]: a webmaster has to choose the hyperlinks she puts on the webpages she is responsible for in order to maximise their PageRank, hence their visibility on Web search engines. The PageRank is essentially the invariant distribution of a random walker on the graph of hyperlinks [4], which is described a stochastic matrix and therefore equivalent to a consensus problem. While [6,12] propose explicit algorithms to maximize the PageRank of a given page by choosing to rewire the hyperlinks leaving some webpages, we focus on an asymptotic situation where an ever-growing World-WideWeb is called weakly democratic if it is impossible for a fixed small set of webpages to retain a fixed fraction of the total PageRank as more and more new pages are being added, and democratic if the top PageRank of the WWW keeps decreasing to zero as the network grows.

Let us briefly remark about the novelty of these re-

sults and their connections with classical Markov chains theory. As Proposition 5 suggests, there is an intimate connection between weak-democracy of a sequence of stochastic matrices and the non-positive recurrence of the limit infinite stochastic matrix $P^{(\infty)}$ (precisely defined below). However, as Examples 3 and 7 analyzed in Sections 3 and 4 show, neither weak democracy nor democracy is equivalent to the non-positive recurrence of $P^{(\infty)}$. While non-positive recurrence clearly plays an important role in our paper, there is nevertheless no obvious way to deduce our results from classical theory of infinite Markov chains.

We remark that unlike the typical perturbation results available in the literature where it is assumed that $|P_{ij}^{(n)} - \tilde{P}_{ij}^{(n)}|$ are small, here we leave the possibility of large perturbations but localized in a small set. For this type of perturbations, bounds like in [17] (see Theorem 2.1) involving the reciprocal of the spectral gap of the matrix are of little utility since they will typically blow up when the number of nodes goes to ∞. A formula for updating the invariant probability of a Markov chain upon the change of a row of the probability transition matrix has been derived in [16], but there is no easy way to exploit it for an asymptotic behavior.

1.5 Outline of the paper

Section 2 is devoted to introducing all relevant notation, to formulate the problem, to present some relevant examples, and to state the main results. In Section 3 we study a weaker version of democracy when the convergence to 0 of the consensus weights (or invariant probabilities) is only pointwise. We prove Theorem 2 shows that under the only assumptions that the limit chain is irreducible, weak democracy is preserved under local perturbations. Moreover we show with Example 6 that irreducibility of the limit chain is a necessary condition for this type of results. In Section 4 we start analyzing democracy discussing its relation with non-positive recurrence. Our main results are Theorem 6 which characterize democracy in terms of the lack of positively recurrent states in the asymptotic limits of the sequence, and Corollary 7 which establishes the preservation of democracy under local perturbation. Finally, Section 5 focuses on time-reversible chains and gives an application of Corollary 7. The main result is Theorem 3 which guarantees that democracy is preserved under local perturbations even when these possibly break the time-reversibility of the chain.

This paper extends a preliminary partial version that has appeared in conference proceedings [7]. The main definitions and two main results of this paper, Theorems 2 and 3, were already stated in [7]. However in this paper we present complete and more elegant proofs, thanks in part to the new concepts of Section 4. Example 6 and Figure 2, are reproduced from [7].

2 Assumptions, examples, and main results

2.1 Stochastic matrices and graphs

Given a set V (finite or countably infinite), we denote by $\mathbb{1}_V$ the vector in $\mathbb{R}^V$ having all components equal to 1. A stochastic matrix P on V is any $P \in \mathbb{R}^{V \times V}$ such that $P_{ij} \geq 0$ for all $i, j \in V$ and $P\mathbb{1}_V = \mathbb{1}_V$.

To any stochastic matrix P on a set V (finite or infinite) we can associate a directed transition graph $\mathcal{G} = (V, \mathcal{E})$ on the set of vertices V and where $(i, j) \in \mathcal{E}$ if and only if $P_{ij} > 0$. If $i \in V$, $N(i) = \{j \in V \setminus \{i\} \,|\, (i, j) \in V\}$ denotes the set of out-neighbors of i. Notice that i is never considered in $N(i)$ even when $(i, i) \in \mathcal{E}$. A path on $\mathcal{G}$ is a sequence of vertices $\gamma = (l_1, \ldots, l_M)$ such that $(l_s, l_{s+1}) \in \mathcal{E}$ for all $s = 1, \ldots, M - 1$. We say that γ starts from l_1 and ends in l_M, or, also, that joins l_1 to l_M. The length of γ is denoted $l(\gamma) := M - 1$. The graph $\mathcal{G}$ is said to be strongly connected if any pair of vertices can be joined by a path; in this case the matrix P generating the graph is called irreducible. We will denote by $d_{\mathcal{G}}$ the usual distance on the vertices of a strongly connected graph as the length of a minimal path between vertices. If $W \subseteq V$, we put $\mathcal{G}(W) = (W, \mathcal{E} \cap (W \times W))$.

If P is an irreducible stochastic matrix on a finite set V, it admits a unique vector $\pi \in \mathbb{R}^V$ with $\pi_i > 0$ for all $i \in V$ such that $\sum_i \pi_i = 1$ and $\pi^* P = \pi^*$. If, moreover, P is also aperiodic (see [15] for the exact definition), it follows that $\lim_{n \to +\infty} P^n = \pi \mathbb{1}^*$: in this case, because of the interpretation presented in the Introduction, P is also called a consensus matrix and the relative vector π the consensus weight vector.

Any stochastic matrix $P \in \mathbb{R}^{V \times V}$ can be interpreted as the transition matrix of a Markov chain on V. Given a probability vector $\rho \in \mathbb{R}^V$ ($\rho_i \geq 0$ for all $i \in V$ and $\sum_i \rho_i = 1$), the pair (ρ, P) is called a Markov chain and defines a stochastic process X_t (for times $t \in \mathbb{N}$) taking values in V, called the state space of the Markov chain. The initial state X_0 is distributed according to ρ and the distribution of X_{t+1} conditioned to $X_t = j$ is given by the j-th row of P. This implies that X_t on the state space V is given by $\rho^* P^t$. If ρ is such that $\rho^* P = \rho^*$, it is said to be an invariant probability vector for P, and the corresponding Markov chain is said to be stationary. Consensus weight vectors can thus be interpreted as invariant probabilities. This dual Markov chain interpretation turns out to be a very powerful tool to state, prove and interpret our technical results and will thus be freely used in the following.

Given an undirected graph $\mathcal{G} = (V, \mathcal{E})$ (i.e., $(i, j) \in \mathcal{E}$ iff $(j, i) \in \mathcal{E}$) which is connected, an important class of stochastic matrices generating $\mathcal{G}$ can be constructed starting from a symmetric non-negative valued matrix $C \in \mathbb{R}^{V \times V}$ adapted to $\mathcal{G}$ (i.e., $C_{ij} \neq 0$ iff $(i, j) \in \mathcal{E}$) and

defining the stochastic matrix

$$P_{ij} = \frac{C_{ij}}{C_i} \tag{3}$$

where $C_i = \sum_j C_{ij}$ is assumed to be finite for all $i \in V$. A stochastic matrix of this type is called time-reversible (or simply 'reversible'), while C is called a conductance matrix. If V is finite, the unique invariant probability of P is given by $\pi_i = C_i / \sum_j C_j$. In the case when $C_{ij} \in \{0,1\}$ and $C_{ii} = 0$ for all i, we obtain the matrix described in (2) with $\tau = 0$ which, in the probabilistic jargon, is called the simple random walk on $\mathcal{G}$. Putting instead $C_{ii} = d_i\tau/(1-\tau)$ for all i, we obtain the matrix described in (2): for $\tau \neq 0$ this is called a lazy simple random walk.

2.2 Assumptions and formulation of the problem

We assume we have fixed an infinite universe set $\mathcal{V}$ and a sequence V_n ($n \in \mathbb{N}$) of finite cardinality subsets of $\mathcal{V}$ which is nested (e.g. $V_n \subseteq V_{n+1}$) and is such that $\cup_n V_n = \mathcal{V}$. We then consider a sequence of irreducible stochastic matrices $P^{(n)}$ on the state spaces V_n (and as such of increasing dimension) with a property which essentially establishes that for every node $i \in \mathcal{V}$, the non-zero values of the i-th row of $P^{(n)}$ remain fixed for n sufficiently large. Formally, we assume that, for every $i \in \mathcal{V}$, there exists a positive integer $n_i \in \mathbb{N}$ such that $i \in V_{n_i}$ and

$$P_{ij}^{(n)} = P_{ij}^{(n_i)}\,, \quad \forall n \geq n_i\,,\ \forall j \in V_{n_i} \tag{4}$$

Notice first of all that in the formula above $i, j \in V_{n_i} \subseteq V_n$ for $n \geq n_i$ and thus it makes perfect sense to consider $P_{ij}^{(n)}$: the i-th rows $P^{(n)}$ and $P^{(n_i)}$ are vectors of different lengths only differing by zero entries. Therefore (4) means that the ith row of $P^{(n)}$ remains constant as soon as $n \geq n_i$, except for a growing string of zero entries. If we consider the associated graphs $\mathcal{G}^{(n)}$ we have in particular that the out-neighbor s of node i remain the same in all graphs for $n \geq n_i$. Property (4) allows us to define, in a natural way, a limit stochastic matrix on $\mathcal{V}$. For every $i, j \in \mathcal{V}$, we define

$$P_{ij}^{(\infty)} = \begin{cases} P_{ij}^{(n_i)} & \text{if } j \in V_{n_i} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Throughout the paper we will always use the following notation convention. All quantities related to the stochastic matrix $P^{(n)}$ (including $n = \infty$) will have the superscript (n): $\pi^{(n)}$ is the invariant probability (uniquely defined for $n < \infty$ as we assume irreducibility of $P^{(n)}$), $\mathcal{G}^{(n)}$ the associated graph, $N^{(n)}(i)$ the out-neighbor set of i in $\mathcal{G}^{(n)}$. Notice that, by (4), $\sum_{j \in V_{n_i}} P_{ij}^{(n)} = 1$ for every $n \geq n_i$. This implies that $N^{(n)}(i) = N^{(n_i)}(i)$ for every $n \geq n_i$. In particular, $N^{(\infty)}(i)$ is finite for all $i \in \mathcal{V}$.

Definition 1 *The sequence of stochastic matrices $P^{(n)}$ is said to be:*

- *weakly democratic if for all $i \in \mathcal{V}$, $\pi_i^{(n)} \to 0$ for $n \to +\infty$.*
- *democratic if $||\pi^{(n)}||_\infty := \max\limits_{i \in V_n} \pi_i^{(n)} \to 0$ for $n \to +\infty$.*

In this paper we want to investigate the preservation of the properties expressed in Definition 1, under finite perturbations. More precisely, we fix a finite subset $W \subseteq V_1$ and another sequence of irreducible stochastic matrices $\tilde{P}^{(n)}$ on V_n such that

$$\begin{aligned} \tilde{P}_{ij}^{(n)} &= P_{ij}^{(n)} \quad \forall i \in V_n \setminus W\,,\ \forall j \in V_n \\ \tilde{P}_{ij}^{(n)} &= \tilde{P}_{ij}^{(1)} \quad \forall\, i \in W\,,\ \forall j \in V_1 \end{aligned} \tag{6}$$

In other terms, $\tilde{P}^{(n)}$ can be seen as a perturbed version of $P^{(n)}$ with the perturbation confined to the fixed subset W and stable (it does not change as n increases). Notice that $\tilde{P}^{(n)}$ satisfy the same stabilization assumption (4) than $P^{(n)}$, and thus, also for this perturbed sequence we can define, following (5), the asymptotic chain $\tilde{P}^{(\infty)}$. The assumptions $W \subseteq V_1$ and the second one in (6) are taken for simplicity. The crucial fact needed is that W is finite and that for every $i \in W$ and $j \in \mathcal{V}$, $\tilde{P}_{ij}^{(n)}$ becomes constant for large n.

2.3 Examples

A general and fundamental example fitting in the formalism of this section can be obtained by starting from an infinite graph and considering simple random walks on larger and larger finite subgraphs of it. Precisely, consider an infinite connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that each vertex $i \in \mathcal{V}$ has a finite degree d_i. Consider a nested sequence V_n of finite cardinality subsets of $\mathcal{V}$ such that $\cup_n V_n = \mathcal{V}$. Assume that the subgraphs $\mathcal{G}^{(n)} = \mathcal{G}(V_n) = (V_n, \mathcal{E}^{(n)})$ (where, we recall, $\mathcal{E}^{(n)} = \mathcal{E} \cap V_n \times V_n$) are connected. Notice that $\mathcal{G}^{(\infty)} = \mathcal{G}$. Consider the lazy simple random walk (2) on it: for $i, j \in V_n$ we put $P_{ij}^{(n)} = \frac{1-\tau}{d_i^{(n)}}$ for $(i,j) \in \mathcal{E}^{(n)}$ with $i \neq j$ and $P_{ii}^{(n)} = \tau$. According to the notation agreement, $d_i^{(n)}$ denotes the degree of node i in $\mathcal{G}^{(n)}$: clearly for sufficiently large n, this degree coincides with d_i. Notice that the invariant probability $\pi^{(n)}$ of $P^{(n)}$, is such that

$$||\pi^{(n)}||_\infty \leq \frac{d^{(n)}}{|\mathcal{E}^{(n)}|} \tag{7}$$

where $d^{(n)} := \sup_{i\in V_n} d_i$. In particular, this shows that if $d^{(n)} = o(|\mathcal{E}^{(n)}|)$ for $n \to +\infty$, the sequence $P^{(n)}$ is democratic.

The condition $d^{(n)} = o(|\mathcal{E}^{(n)}|)$ is verified in all case s where the original graph has bounded degrees (e.g. d-dimensional lattices and more general regular graphs). Another case are the random geometric graphs: indeed at the connectivity threshold, maximal degrees grow logarithmically with the number of nodes, with high probability [19].

Example 1 *Consider the d-dimensional infinite lattice over $\mathcal{V} = \mathbb{Z}^d$ formally defined as follows. Consider the canonical basis vectors $e_i \in \mathbb{Z}^d$ for $i = 1, \dots, d$ and put $\Lambda = \{\pm e_i \mid i = 1, \dots, d\}$. Then we define $\mathcal{G} = (\mathbb{Z}^d, \mathcal{E})$ where $\mathcal{E} := \{(v, w) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid v - w \in \Lambda\}$. Consider $V_n = [-n, n]^d$. $\mathcal{G}^{(n)} = \mathcal{G}(V_n)$ is the d-dimensional grid with $2n + 1$ nodes in each direction. Internal nodes have degree $2d$ while boundary nodes have degrees in $\{1, 2, \dots, d\}$. In particular, $d^{(n)} = d$ for all n. This says that the corresponding simple random walks on such grid graphs form a democratic sequence.*

A more general setting is obtainable by replacing the simple random walks with more general time-reversible matrices. Precisely, in the same graph setting proposed above, assume to have fixed a sequence of conductance matrices $C^{(n)}$ adapted to $\mathcal{G}^{(n)}$ such that

(a) for every $i \in \mathcal{V}$, there exist $n_i \in \mathbb{N}$ such that $i \in V_{n_i}$ and

$$C^{(n)}_{ij} = C^{(n_i)}_{ij}\,, \quad \forall n \geq n_i\,,\ \forall j \in V_{n_i} \tag{8}$$

(b) there exist constants $0 < a < b$ such that

$$a < C^{(n)}_{ij} < b\,, \quad \forall (i, j) \in \mathcal{E}^{(n)}\,,\ \forall n \in \mathbb{N} \tag{9}$$

Let $P^{(n)}$ be the time-reversible stochastic matrix on V_n associated with $C^{(n)}$ in the sense of (3). Notice that $P^{(n)}$ is irreducible and satisfies the stabilization condition (4). Notice also that $P^{(\infty)}$ is time-reversible and coincides with the stochastic matrix associated with the limit of conductances $C^{(\infty)}$. A simple check on the invariant probabilities shows that the condition $d^{(n)} = o(|\mathcal{E}^{(n)}|)$ for $n \to +\infty$ remains sufficient for democracy.

The following is instead a possible way to construct non time-reversible examples, as they occur on graphs that are not undirected.

Example 2 *Consider the following modification of the grid graphs considered in Example 1. Similarly, we consider $\mathcal{V} = \mathbb{Z}^d$, the sequence of subsets $V_n = [-n, n]^d$, and the subset $\Lambda^+ = \{e_i \mid i = 1, \dots, d\}$, where e_i is the elementary vector $(0, \dots, 0, 1, 0, \dots, 0)$ with a single 1 entry in ith position. We then define $\mathcal{G}^{(n)} = (V_n, \mathcal{E}^{(n)})$ where $\mathcal{E}^{(n)} := \{(v, w) \in [-n, n]^d \times [-n, n]^d \mid v - w \in \Lambda^+\}$ where the $-$ operation is to be interpreted modulo $2n + 1$. $\mathcal{G}^{(n)}$ is a d-dimensional grid with all directed edges without boundary (shaped like a torus). It is clearly non time-reversible as the walker may for instance jump from node 0 to any node e_i in one step, then to any node $e_i + e_j$, but not back to 0. It is easy to realize that $\mathcal{G}^{(\infty)}$ is the infinite lattice $\mathbb{Z}^d$ with all directed edges. In the trivial case $d = 1$, $\mathcal{G}^{(n)}$ is simply the $2n + 1$-node directed cycle where the random walker runs always in the same direction, and $\mathcal{G}^{(\infty)}$ is the infinite directed path whose nodes are indexed by $\mathbb{Z}$. If we consider the simple random walk $P^{(n)}$ on $\mathcal{G}^{(n)}$, we have that the invariant probability is always the uniform one, so that the sequence is democratic. Modifying the structure subset Λ^+, it is possible to construct a whole variety of graphs named Abelian Cayley graphs for which similar considerations apply.*

Both examples above lead to democratic sequence of stochastic matrices. We now present examples of non weakly democratic sequence of stochastic matrices and also weakly democratic sequences which are not democratic.

Example 3 *Let $V_n = \{1, \dots, n\}$, $0 < \delta < 1$, and*

$$P^{(n)}_{ij} := \begin{cases} \delta & \text{if } i < n\,,\ j = i + 1 \\ 1 - \delta & \text{if } i > 1\,,\ j = i - 1 \\ 1 - \delta & \text{if } i = j = 1 \\ \delta & \text{if } i = j = n \end{cases}$$

It is possible to verify that, for $\delta \neq 1/2$, the invariant probability measure is given by

$$\pi^{(n)}_i = \left(\frac{\delta}{1-\delta}\right)^{i-1} \frac{1 - \left(\frac{\delta}{1-\delta}\right)}{1 - \left(\frac{\delta}{1-\delta}\right)^n}$$

If $0 < \delta < 1/2$ it follows that

$$\lim_{n\to+\infty} \pi^{(n)}_i = \left(\frac{\delta}{1-\delta}\right)^{i-1} \left(1 - \left(\frac{\delta}{1-\delta}\right)\right)$$

so that the stochastic matrix is not weakly democratic. Consider now the case when $1/2 < \delta < 1$. Then,

$$\lim_{n\to+\infty} \pi^{(n)}_i = 0 \quad \forall i$$

However,

$$\lim_{n\to+\infty} ||\pi^{(n)}||_\infty = \lim_{n\to+\infty} \pi^{(n)}_n = 1 - \frac{1-\delta}{\delta}$$

so the stochastic matrix is weakly democratic but not democratic.

2.4 Main results

In this paper we will present two main results. The first one, the Theorem below, is a robustness result of weak democracy.

Theorem 2 *Consider a sequence of weakly democratic irreducible stochastic matrices $P^{(n)}$ satisfying (4) with $P^{(\infty)}$ irreducible. Then, any perturbed sequence $\tilde{P}^{(n)}$ of irreducible stochastic matrices satisfying (6) and such that $\tilde{P}^{(\infty)}$ is irreducible, is also weakly democratic.*

Theorem 2 will be proven in Section 3 where we will also present an example showing that irreducibility of the limit stochastic matrix is an assumption which can not be dropped.

The second result is within the framework of time-reversible stochastic matrices. We have fixed an infinite universe set $\mathcal{V}$, a nested sequence V_n of finite cardinality subsets of $\mathcal{V}$ such that $\cup_n V_n = \mathcal{V}$, a sequence of connected undirected graphs $\mathcal{G}^{(n)} = (V_n, \mathcal{E}^{(n)})$, and a sequence of conductance matrices $C^{(n)}$ adapted to $\mathcal{G}^{(n)}$ satisfying (8). We impose two extra conditions. First, the boundedness of the degrees on the infinite graph $\mathcal{G}^{(\infty)}$:

$$d := \sup_{i\in\mathcal{V}} |N^{(\infty)}(i)| < +\infty\,. \tag{10}$$

The second condition strengthens (9) and is a finiteness condition on the values assumed by the conductance:

$$\Theta := \{C^{(n)}_{ij} \,|\, i,j \in \mathcal{V}\,,\ n\in\mathbb{N}\} \text{ is a finite set} \tag{11}$$

Here is our main result (proof will be presented in Section 5):

Theorem 3 *Consider a sequence of irreducible stochastic matrices $P^{(n)}$ constructed through a sequence of conductance matrices $C^{(n)}$ satisfying (8), (10), and (11). Suppose that $P^{(\infty)}$ is irreducible. Suppose moreover that the subset W and the perturbed sequence $\tilde{P}^{(n)}$ are chosen to satisfy assumptions (6) and $\tilde{P}^{(\infty)}$ is irreducible. Then, the sequence $\tilde{P}^{(n)}$ is democratic.*

The context of application of Theorem 3 is quite wide and includes many of the common cases which show up in consensus problems. We propose a couple of concrete instances below. The first one deals with simple random walks on directed graphs obtained by finitely perturbing a sequence of undirected graphs (e.g cutting some edges in just one direction). The second instead models the presence of a finite community of nodes with homophilic behavior.

Example 4 *Consider an infinite connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that each vertex $i \in \mathcal{V}$ has a finite degree. Assume that the subgraphs $\mathcal{G}^{(n)} = \mathcal{G}(V_n) = (V_n, \mathcal{E}^{(n)})$ (where, we recall, $\mathcal{E}^{(n)} = \mathcal{E} \cap V_n \times V_n$) are connected and consider the lazy simple random walk $P^{(n)}$ described in Section 2.3 $P^{(n)}_{ij} = \frac{1-\tau}{d^{(n)}_i}$ for $(i,j) \in \mathcal{E}^{(n)}$ with $i \neq j$ and $P^{(n)}_{ii} = \tau$. Clearly, (8), (10), and (11) are all satisfied. Fix now any finite subset $W \subseteq V_1$ and consider a strongly connected perturbed graph (possibly no longer undirected) $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ which can only differ from $\mathcal{G}$ for edges outgoing from W: for every $v \in \mathcal{V} \setminus W$ and $v' \in V$, it holds $(v, v') \in \mathcal{E}$ if and only if $(v, v') \in \tilde{\mathcal{E}}$. Assume that the subgraphs $\tilde{\mathcal{G}}^{(n)} = \tilde{\mathcal{G}}(V_n)$ are connected and consider the lazy simple random walk $\tilde{P}^{(n)}$ on $\tilde{\mathcal{G}}^{(n)}$ formally defined as $\tilde{P}^{(n)}_{ij} = \frac{1-\tau}{\tilde{d}^{(n)}_i}$ for $(i,j) \in \tilde{\mathcal{E}}^{(n)}$ with $i \neq j$ and $\tilde{P}^{(n)}_{ii} = \tau$, where $\tilde{d}^{(n)}_i$ is the number of outgoing neighbors of i in the graph $\tilde{G}^{(n)}$. Theorem 3 can be applied to conclude that $\tilde{P}^{(n)}$ is a democratic sequence.*

Example 5 *Let $\mathcal{G}$, $\mathcal{G}^{(n)}$ and $P^{(n)}$ as in the previous example. Given a finite subset $W \subseteq V_1$, define the perturbed sequence $\tilde{P}^{(n)}$ as follows*

$$\tilde{P}^{(n)}_{ij} = \begin{cases} \frac{\lambda(1-\tau)}{d^{(n)}_i + (\lambda-1)d^{(n)}_{i,W}} & \text{for } i \in W,\ j \in W \text{ neighbor of } i \\ \frac{1-\tau}{d^{(n)}_i + (\lambda-1)d^{(n)}_{i,W}} & \text{for } i \in W,\ j \notin W \text{ neighbor of } i \\ \tau & \text{for } j = i \end{cases} \tag{12}$$

where $d^{(n)}_{i,W}$ is the number of neighbors of i inside W, and $\lambda \geq 1$ measures the homophily of community W, measuring a strong mutual influence inside W and weak influence by the agents outside W in an opinion dynamics interpretation.

Theorem 3 says that the minority in W, even if their homophily parameter λ is very large, cannot unilaterally break the democracy: their consensus weights $\tilde{\pi}_i$ will go to 0 in the large scale limit and therefore also their opinion, regardless of their conservativeness, will have a negligeable impact in the asymptotic consensus opinion of the global population. See illustration on Figure 1.

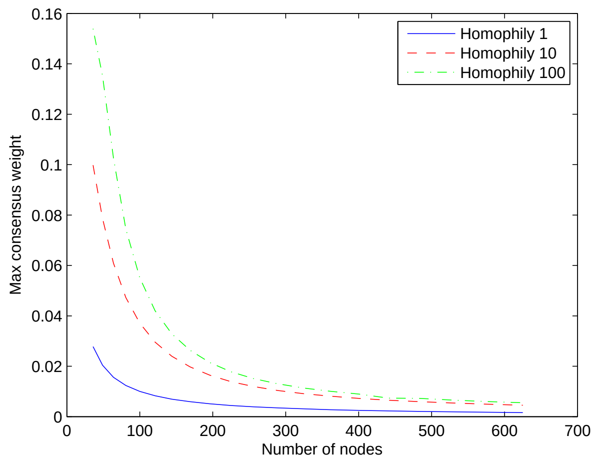

Fig. 1. We illustrate our main result, Theorem 3, on Example 4 applied to a bidimensional torus grid, whose node set is of the form $[-n,n]^2$, as in Example 2. We consider a lazy random walk with self-confidence parameter $\tau = 0.1$. We then modify the behavior of the 9-node community $W = [-1,1]^2$, with a homophily parameter $\lambda = 1$, 10 or 100. We plot the maximum consensus weight $\|\pi^{(n)}\|_\infty$ as a function of the number of nodes. Note that homophily $\lambda = 1$ corresponds to the unperturbed lazy random walk, where all nodes have equal weight. A high homophily creates a community W of nodes of much higher consensus weight than the other nodes. As predicted by Theorem 3, this homophilic community, despite breaking time-reversibility of the random walk, does not break democracy. As the number of nodes increases indeed, the maximum weight converges to zero.

3 Weakly democratic stochastic matrices and hitting times

In this section we prove Theorem 2. Techniques employed, as it will happen in the following sections, are essentially probabilistic. The key point is the connection between weak democracy and hitting and return times of the underlying Markov chain.

For the sake of completeness, we briefly recall below a number of standard concepts of Markov chains which will play an important role in the following analysis. Recall from Section 2 that a Markov chain on a state space V is a stochastic process (X_t) described by an initial probability vector $\rho \in \mathbb{R}^V$ describing the distribution of X_0 and a stochastic matrix $P \in \mathbb{R}^{V\times V}$ describing the transition probabilities from X_t to X_{t+1}. We will denote by $\mathbb{P}_i$ the probability relative to such a process when $\rho = \delta_i$ the delta probability measure concentrated on i. Similarly we denote by $\mathbb{E}_i$ the corresponding mean operator. If $S \subseteq V$, τ_S and τ_S^+ denote, respectively, the first hitting time and the first return time into S:

$$\begin{aligned}\tau_S &:= \min\{t \geq 0 \,|\, X_t \in S\}\\ \tau_S^+ &:= \min\{t \geq 1 \,|\, X_t \in S\} \;\; (X_0 \in S)\,.\end{aligned}$$

If $S = \{i\}$ we use the notation τ_i and τ_i^+ for τ_S and τ_S^+, respectively. For notation simplicity we will use the notation E_{ij} for $\mathbb{E}_i(\tau_j)$ and E_{i+} for $\mathbb{E}_i(\tau_i^+)$. In the case when P is irreducible and $\pi \in \mathbb{R}^V$ is its corresponding invariant probability, we have the following remarkable relation: $\pi_i = (E_{i+})^{-1}$ for all $i \in V$ (see for instance [3,15]).

We now assume we have fixed a sequence of stochastic matrices $P^{(n)}$ and a perturbed one $\tilde{P}^{(n)}$ as in Section 2.2. According to our general terminological assumptions we will denote the above quantities with a superscript (n) when referred to $P^{(n)}$ and with in addition a tilde on top when referred to the perturbed one (e.g. $E_{ij}^{(n)}$, $E_{i+}^{(n)}$, $\tilde{E}_{ij}^{(n)}$, $\tilde{E}_{i+}^{(n)}$).

Since $\pi_i^{(n)} = (E_{i+}^{(n)})^{-1}$, it follows that a sequence of stochastic matrices $P^{(n)}$ is democratic if and only if $\lim_{n\to\infty} E_{i+}^{(n)} = \infty$ for all i.

The following theorem is proved in [7].

Proposition 4 *[7] For a sequence of irreducible stochastic matrices $P^{(n)}$ satisfying (4) and such that $P^{(\infty)}$ is irreducible, the following conditions are equivalent:*

(a) *The sequence is weakly democratic.*
(b) *There exists $j \in \mathcal{V}$ such that*

$$E_{j+}^{(n)} \to +\infty \;\text{ for } n \to +\infty$$

(c) *There exist $j \in \mathcal{V}$ and a finite subset $Y \subseteq \mathcal{V} \setminus \{j\}$ such that*

$$\max_{i\in Y} E_{ij}^{(n)} \to \infty \;\text{ for } n \to \infty$$

(d) *For every $j \in \mathcal{V}$ there exists a finite subset $Y \subseteq \mathcal{V} \setminus \{j\}$ such that*

$$\max_{i\in Y} E_{ij}^{(n)} \to \infty \;\text{ for } n \to \infty$$

Remark: It follows from Proposition 4 that when weak democracy fails, then $\pi_i^{(n)}$ remains bounded away from 0 for all i. This phenomenon is essentially due to the stabilizing condition (4) which prevents nodes' degrees to blow to ∞. See [11] for examples where instead this condition is not imposed.

We are now ready to prove Theorem 2.

Proof (of Theorem 2) We first prove it in the case when $W = \{w\}$, i.e., when only one node w is perturbed.

Applying Proposition 4 (condition (d)) to $P^{(n)}$ we have that there exists $Y \subseteq \mathcal{V} \setminus \{w\}$ such that

$$\max_{i \in Y} E_{iw}^{(n)} \to \infty \text{ for } n \to \infty$$

Notice that $E_{iw}^{(n)} = \tilde{E}_{iw}^{(n)}$ for every $i \in Y$. Hence, condition (c) of Proposition 4 is verified for $\tilde{P}^{(n)}$. Hence $\tilde{P}^{(n)}$ is weakly democratic.

Consider now a general, finite set of perturbed nodes W. The idea is to use induction on the cardinality of W. However, some attention must be paid to the possible loss of irreducibility of $P^{(\infty)}$ during the inductive path. To overcome this, we consider three cases.

(1) Assume that for every node $w \in W$, the set of outgoing edges of w in the graph of $P^{(\infty)}$ is included in the set of outgoing edges of w in the graph of $\tilde{P}^{(\infty)}$. For every $W' \subseteq W$, construct the sequence of stochastic matrices $P_{W'}^{(n)}$, obtained from $P^{(n)}$ by replacing every row corresponding to $w \in W'$ by the corresponding row of $\tilde{P}^n$. In particular, $P_W^{(n)} = \tilde{P}^{(n)}$. Then, obviously every $P_{W'}^{(\infty)}$ is irreducible. A straightforward inductive procedure applied to a sequence of W' where one node is added at a time now allows to prove that $\tilde{P}^{(n)}$ is weakly democratic.

(2) Now assume that for every node $w \in W$, the set of outgoing edges of w in the graph of $P^{(\infty)}$ contains the set of outgoing edges of w in the graph of $\tilde{P}^{(\infty)}$. Then every $P_{W'}^{(\infty)}$ is irreducible again and, arguing as in previous case, we conclude that $\tilde{P}^{(n)}$ is weakly democratic.

(3) If none of the above apply, consider the intermediate sequence of chains $Q^{(n)} = \frac{1}{2}(P^{(n)} + \tilde{P}^{(n)})$. Then the first case above applies to $P^{(n)}$ and $Q^{(n)}$, showing that Q^n is weakly democratic. Now the second case applied to $Q^{(n)}$ and $\tilde{P}^{(n)}$ shows that $\tilde{P}^{(n)}$ is weakly democratic. ■

We give now an example of a weakly democratic family of stochastic matrices $P^{(n)}$, converging to an irreducible infinite stochastic matrix, such that, modifying the transition probabilities from just one state, one gets a non weakly democratic family. This is not in contradiction with Theorem 2, because this perturbed family converges to a reducible infinite stochastic matrix. This shows that the irreducibility assumption on $\tilde{P}^\infty$ is required.

Example 6 *The chains $P^{(n)}$ and $\tilde{P}^{(n)}$ are defined as in Figure 2 (notice that $P^{(n)}$ fits in Example 2). It is clear that the stationary distribution on states of $P^{(n)}$ is uniform, therefore $P^{(n)}$ is weakly democratic. We shall now show that the sequence of chains $\tilde{P}^{(n)}$ is not weakly democratic.*

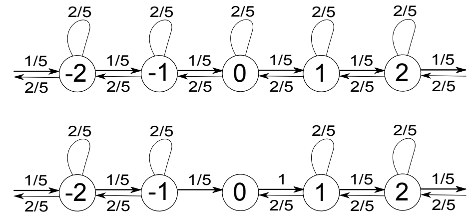

Fig. 2. The chains of Example 6. *Top*: the chain $P^{(\infty)}$. The chain $P^{(n)}$ is built from $P^{(\infty)}$ by identifying $\lfloor n/2 \rfloor$ and $-\lceil n/2 \rceil$, thus making $P^{(n)}$ a cyclic chain. *Bottom*: the chain $\tilde{P}^{(\infty)}$, which differs from $P^{(\infty)}$ only by transitions from state 0. The chains $\tilde{P}^{(n)}$ differ from $P^{(n)}$ in the same way.

Indeed $\tilde{E}_{0+}^{(n)} = 1 + \tilde{E}_{10}^{(n)} = 1 + E_{10}^{(n)}$. The last equality follows from the fact that computing the first hitting time from 1 to 0 does not require the knowledge of the transition probabilities from 0. Moreover, $E_{10}^{(n)} = E_{1,\{0,n-1\}}^{(\infty)}$, by construction of $P^{(n)}$. Note that evaluating $E_{1,\{0,n-1\}}^{(\infty)}$ is equivalent to the classical gambler's ruin problem (random walk with a drift and two absorbing barriers) [8]. It follows that $E_{1,\{0,n-1\}}^{(\infty)} < c$, where c is a finite constant, independent of n. Therefore, $\tilde{\pi}_0^{(n)} > 1/c$. Thus, the sequence $\tilde{P}^{(n)}$ is not weakly democratic. With the same technique, one may show that the probability $\tilde{\pi}_k^{(n)}$ for vertices $k > 0$ does not converge to zero, while $\tilde{\pi}_k^{(n)} \to 0$ for any $k < 0$.

4 Weak democracy, democracy and recurrence

There are important connections between weak democracy and democracy of a sequence $P^{(n)}$ and the positive recurrence of the limit stochastic matrix $P^{(\infty)}$. We recall that a vertex i is said to be positively recurrent if $E_{i+}^{(\infty)} < +\infty$ [20]. If the matrix is irreducible and one vertex is positively recurrent, then all vertices are positively recurrent and we say, in this case, that the chain is positively recurrent. We have the following simple relation:

Proposition 5 *Consider a sequence of irreducible stochastic matrices $P^{(n)}$ such that both $P^{(n)}$ and its transpose $(P^{(n)})^*$ satisfy (4). Assume moreover that $P^{(\infty)}$ does not have positively recurrent vertices. Then, the sequence $P^{(n)}$ is weakly democratic.*

Proof The proof follows by standard arguments. Indeed, a diagonal argument shows that we can always find a subsequence n_k such that $\pi_i^{(n_k)}$ converges to some value that we denote $\pi_i^{(\infty)}$ for all $i \in \mathcal{V}$ and a dominated convergence argument shows that $\pi^{(\infty)*} P^{(\infty)} = \pi^{(\infty)*}$ (where the $.^*$ denotes the matrix transposition). Now,

if $P^{(n)}$ is not weakly democratic, $\pi_i^{(\infty)}$ is different from 0 for all i. Hence, $P^{(\infty)}$ admits an invariant probability measure and thus $E_{i+}^{(\infty)} < +\infty$. ■

Let us now go back to Example 3 to show that non positive recurrence does not yield democracy. Indeed $P^{(\infty)}$ in that case is given by

$$P_{ij}^{(\infty)} := \begin{cases} \delta & \text{if } j = i+1 \\ 1-\delta & \text{if } i > 1\,,\ j = i-1 \\ 1-\delta & \text{if } i = j = 1 \end{cases}$$

It is straightforward to verify that, for $0 < \delta < 1/2$, $P^{(\infty)}$ admits an invariant probability measure given by

$$\pi_i^{(\infty)} = \left(\frac{\delta}{1-\delta}\right)^{i-1}\left(1 - \left(\frac{\delta}{1-\delta}\right)\right)$$

Therefore the chain is positively recurrent. Instead, if $1/2 < \delta < 1$, $P^{(\infty)}$ does not admit any invariant probability measure, the chain is thus non positively recurrent. Nevertheless, the sequence $P^{(n)}$ is only weakly democratic and not democratic.

We now present another example showing that $P^{(n)}$ may well be democratic while $P^{(\infty)}$ is positively recurrent. The reason is that there can be 'boundary effects' coded in the sequence $P^{(n)}$ which disappear in the limit , in the sense that the consensus weight vector of $P^{(n)}$ is largely determined by the many entries in $P^{(n)}$ that have not yet stabilized to their eventual value in $P^{(\infty)}$, as we illustrate in the next example.

Example 7 *Fix an infinite universe set $\mathcal{V}$, a nested sequence $V_n = \{1, \dots, n\}$ of finite cardinality subsets of $\mathcal{V} = \mathbb{N}$, a sequence of strongly connected graphs $\mathcal{G}^{(n)} = (V_n, \mathcal{E}^{(n)})$ equipped with simple random walks $P^{(n)}$ for which the usual stability condition applies. Independently from $P^{(n)}$ being democratic or not, we now show that we can modify it to make it democratic without changing the limit matrix $P^{(\infty)}$. Consider the modified sequence of graphs $\tilde{\mathcal{G}}^{(n)} = (V_n \cup A_n, \tilde{\mathcal{E}}^{(n)})$ where $A_n = \{n+1, \dots, n+M_n\}$ and $\tilde{\mathcal{E}}^{(n)} = \mathcal{E}^{(n)} \cup \{(n, n+1), (n+1, n+2), \dots, (n+M_n, 1)\}$. The sequence M_n will be chosen later. Let $\tilde{P}^{(n)}$ be the simple random walk on $\tilde{\mathcal{G}}^{(n)}$. For every $i \in V_n$, let $\gamma_i^{(n)}$ be any simple path in $\mathcal{G}^{(n)}$ connecting i to n and let $q_i^{(n)}$ be the product of probability of the various edges composing the path. Starting from a vertex $i \in V_n$, with probability $q_i^{(n)}$ we will be following path $\gamma_i^{(n)}$ to n and with probability $1/(d_n+1)$ we will then choose the edge connecting to $n+1$. At that point we will be forced to follow the whole directed cycle of length M_n up to 1. Hence,*

$$\tilde{\mathbb{E}}_i^{(n)}(\tau_i^+) \geq \frac{q_i^{(n)} M_n}{d_n + 1} \tag{13}$$

Similar considerations show that, if $i \in A_n$, for sure we have

$$\tilde{\mathbb{E}}_i^{(n)}(\tau_i^+) \geq M_n \tag{14}$$

If we choose M_n to be

$$M_n := \frac{d_n + 1}{(\min\limits_{i \in V_n} q_i^{(n)})} n$$

we obtain from (13) and (14) that

$$\min_{i \in V_n \cup A_n} \tilde{\mathbb{E}}_i^{(n)}(\tau_i^+) = +\infty$$

so that the sequence of modified chains is indeed democratic. Notice that $P^{(\infty)} = \tilde{P}^{(\infty)}$ (nodes in A_n disappear in the limit). If we have started from a positively recurrent $P^{(\infty)}$ (one can consider for instance the sequence in Example 3 with $0 < \delta < 1/2$) we have now constructed an example of a democratic sequence whose limit chain is positively recurrent.

The negative results presented above show that the limit chain by itself is not sufficient to capture the property of democracy of a chain. The reason, as already noticed, are the 'border' effects which disappear in the limit. However, as will be shown below, the non positive recurrence will play a role once we consider different limit notions for the sequence of chains, to include border effects.

We now introduce a concept which will play a central role in the following. We start fixing some notation. If $\mathcal{G} = (V, \mathcal{E})$ is a graph, $i \in V$ and $R > 0$ denote $B_{\mathcal{G}}(i, R) = \{j \in V \mid d_{\mathcal{G}}(i, j) \leq R\}$. For simplicity we will use the notation $\mathcal{G}(i, R) = \mathcal{G}(B_{\mathcal{G}}(i, R))$.

Given a sequence of irreducible stochastic matrices $P^{(n)}$ satisfying (4), we say that a stochastic matrix Q on an infinite set $\mathcal{Z}$ is a *limit* of $P^{(n)}$ if there exist s a subsequence of positive integers n_k, a sequence $l_k \in V_{n_k}$, and a sequence of injective maps $\lambda_k : V_{n_k} \to \mathcal{Z}$, such that

(a) $\lambda_k(l_k) = z_0$ constant.
(b) For every $z \in \mathcal{Z}$, there exists $k_0 \in \mathbb{N}$ such that $z \in \lambda_k(V_{n_k})$ for all $k \geq k_0$.
(c) For every integer $R > 0$ there exists k_0 such that: for every $k \geq k_0$, for every $i, j \in B_{\mathcal{G}^{(n_k)}}(l_k, R)$, we have that
$$P_{ij}^{(n_k)} = Q_{\lambda_k(i)\lambda_k(j)} \tag{15}$$

Notice that properties above essentially assert that the sequence of graphs $\mathcal{G}^{(n_k)}$ become s stable in an arbitrary

fixed neighborhood subgraph of l_k and isomorphic to $\mathcal{Z}$ in a neighborhood of z_0. Moreover, $P^{(n_k)}$ and Q are equal for large k in such neighborhood subgraphs.

Clearly, $P^{(\infty)}$ is always a limit of $P^{(n)}$: it is sufficient to choose any constant sequence l_n. Such a limit point is called *trivial*.

The sequence l_k appearing in the definition above is called a *stabilizing* sequence and is said to be trivial if constant.

For example, consider $P^{(n)}$ describing the simple random walk on the square grid $V_n = [-n, n]^2$, as in Example 1. The trivial limit is the simple random walk on $\mathbb{Z}^2$. However, choosing l_n as the node of coordinate $(0, -n)$ leads to a limit stochastic matrix Q that describes the random walk on $\mathbb{Z} \times \mathbb{N}$, while $l_n = (-n, n)$ leads to a simple random walk on $\mathbb{N} \times \mathbb{N}$.

$P^{(n)}$ will be called *complete* if for any sequence $l_n \in V_n$, there always exists a stabilizing subsequence.

We have the following result which generalizes Proposition 5.

Theorem 6 *Consider a sequence of irreducible stochastic matrices $P^{(n)}$ satisfying (4) which is weakly democratic, complete, and such that every non-trivial limit does not contain positively recurrent points. Then, the sequence $P^{(n)}$ is democratic.*

Proof Let $\pi^{(n)}$ be the invariant probability of $P^{(n)}$. Suppose by contradiction that

$$\lim_{n\to+\infty} \sup_{i\in V_n} \pi_i^{(n)} > 0$$

Then, there exists a subsequence of the positive integers n_k and a sequence $i_k \in V_{n_k}$ such that

$$\pi_{i_k}^{(n_k)} \geq \alpha > 0 \quad \forall k \in \mathbb{N} \tag{16}$$

If i_k admits a constant subsequence (e.g. $i_k = l$ for infinite values of k, then (16) would violate weak democracy. Therefore, by completeness, we can assume with no lack of generality that i_k is stabilizing and non trivial. This guarantees the existence of a sequence of embeddings $\lambda_k : V_{n_k} \to \mathcal{Z}$ and a stochastic matrix Q on $\mathcal{Z}$ satisfying properties (a), (b), and (c) just above. Let $\mathcal{H}$ be the graph induced by the matrix Q. Fix $R > 0$ and consider the events

$$L_k(R) = \{\tau_{i_k}^+ < \tau_{B_{\mathcal{G}^{(n_k)}}(i_k,R)^c}\}$$

$$\Lambda(R) = \{\tau_{z_0}^+ < \tau_{B_{\mathcal{H}}(z_0,R)^c}\}$$

(where the notation A^c denotes the complementary of subset A). Choose now k large enough so that property (b) holds true for every $z \in B_{\mathcal{H}}(z_0, R)$ and also property (c) holds. This implies that for such k, the two stochastic processes governed by $P^{(n_k)}$ and by Q, respectively, and starting from i_k and z_0, respectively, have the same statistics (through the embedding λ_k) as long they are in the balls $B_{\mathcal{G}^{(n_k)}}(i_k, R)$ and $B_{\mathcal{H}}(z_0, R)$, respectively. This yields

$$\begin{aligned}\mathbb{E}_{i_k}^{(n_k)}(\tau_{i_k}^+) &\geq \mathbb{E}_{i_k}^{(n_k)}(\tau_{i_k}^+ \mathbb{1}_{L_k(R)}) + R\mathbb{P}_{i_k}^{(n_k)}(L_k(R)^c) \\ &= \mathbb{E}_{z_0}^Q(\tau_{z_0}^+ \mathbb{1}_{\Lambda(R)}) + R\mathbb{P}_{z_0}^Q(\Lambda(R)^c)\end{aligned} \tag{17}$$

where $\mathbb{P}_{z_0}^Q$ and $\mathbb{E}_{z_0}^Q$ denote probability and mean with respect to the chain Q starting from z_0. From (16) and (17) (recalling that $\pi_{i_k}^{(n_k)} = \mathbb{E}_{i_k}^{(n_k)}(\tau_{i_k}^+)^{-1}$) we then obtain

$$\mathbb{E}_{z_0}^Q(\tau_{z_0}^+ \mathbb{1}_{\Lambda(R)}) + R\mathbb{P}_{z_0}^Q(\Lambda(R)^c) \leq \alpha^{-1}, \ \forall R > 0 \tag{18}$$

Observing that, for $R \to +\infty$, $\Lambda(R) \uparrow \{\tau_{z_0}^+ < +\infty\}$, we obtain that

$$\lim_{R\to+\infty} \mathbb{P}_{z_0}^Q(\Lambda(R)^c) = \mathbb{P}_{z_0}^Q(\tau_{z_0}^+ = +\infty) \tag{19}$$

$$\lim_{R\to+\infty} \mathbb{E}_{z_0}^Q(\tau_{z_0}^+ \mathbb{1}_{\Lambda(R)}) = \mathbb{E}_{z_0}^Q(\tau_{z_0}^+ \mathbb{1}_{\tau_{z_0}^+ < +\infty}) \tag{20}$$

From (19) and (18) we obtain that $\mathbb{P}_{z_0}^Q(\tau_{z_0}^+ = +\infty) = 0$. Considering (20) and using again (18) we finally obtain $\mathbb{E}_{z_0}^Q(\tau_{z_0}^+) < +\infty$, namely that z_0 is positively recurrent. This contradicts the standing assumptions of the theorem. The result is thus proven. ■

A slight variation of the argument used to prove Theorem 6 allows to prove a perturbation result.

Corollary 7 *Consider a sequence of weakly democratic irreducible stochastic matrices $P^{(n)}$ satisfying (4) and such that $P^{(\infty)}$ is irreducible, the sequence is complete, and every non-trivial limit chain does not contain positively recurrent points. Suppose moreover that the subset W and the perturbed sequence $\tilde{P}^{(n)}$ are chosen to satisfy assumptions (6) and $\tilde{P}^{(\infty)}$ is also irreducible. Then, the sequence $\tilde{P}^{(n)}$ is democratic.*

Proof Notice first of all that because of Theorem 2, $\tilde{P}^{(n)}$ is weakly democratic. Notice now that all non-trivial limit chains of $\tilde{P}^{(n)}$ coincide with the non-trivial limits chains of $P^{(n)}$. Hence result follows from Theorem 6. ■

5 Perturbation of time-reversible democratic chains

In this section we present some applications of Corollary 7. In particular, we will prove Theorem 3.

We start with the following result:

Proposition 8 *Suppose (8), (10), and (11) hold true. Then, the corresponding sequence $P^{(n)}$ is complete and all its limit chains are non positively recurrent chains.*

Proof Recall the notation $\mathcal{G}(i,R)$ to denote the subgraph of $\mathcal{G}$ consisting of those vertices whose distance from i is not greater than R. If C is a conductance matrix on $\mathcal{G}$, we denote by $C(i,R)$ its restriction to $B_{\mathcal{G}}(i,R)$.

Consider, preliminarily, the set $\Omega_{d,R,\Theta}$ of all triples $(\mathcal{G},C,x)$ where $\mathcal{G}=(V,\mathcal{E})$ is an undirected graph with degrees bounded from above by d, C is a conductance matrix adapted to $\mathcal{G}$ and taking values in Θ, and $x\in V$ is such that $d_{\mathcal{G}}(x,v)\leq R$ for every $v\in V$. In particular, the cardinality of the vertices of such graphs is bounded from above by d^R. On $\Omega_{d,R,\Theta}$ we can introduce a notion of isomorphism: $(\mathcal{G},C,x)$ and $(\mathcal{H},C',y)$ are called isomorphic (denoted $(\mathcal{G},C,x)\sim(\mathcal{H},C',y)$) if there exists a graph isomorphism $\psi:\mathcal{G}\to\mathcal{H}$ such that $C_{ij}=C'_{\psi(i),\psi(j)}$ for all i and j, and, moreover, $\psi(x)=y$. It is evident (recall that Θ is a finite set) that $\sim$ is an equivalence relation and that the set of equivalence classes $\Omega_{d,R,\Theta}/\sim$ is finite.

Notice now that for any fixed sequence $l_n\in V_n$, and for every positive number R, $(\mathcal{G}^{(n)}(l_n,R),C^{(n)},l_n)\in\Omega_{d,R,\Theta}$. Hence, there exists a subsequence $l_{n_k^R}$ such that $(\mathcal{G}^{(n_k^R)}(l_{n_k^R},R),C^{(n_k^R)}(l_{n_k^R},R),l_{n_k^R})$ all belong to the same equivalence class. Denote by $(\mathcal{H}^{(R)},D^{(R)},z^{(R)})$ a fixed representative in $\Omega_{d,R,\Theta}$ of such class. A straightforward inductive argument shows that the subsequences n_k^R can be chosen in such a way that, if $R_1<R_2$, then $n_k^{R_2}$ is a subsequence of $n_k^{R_1}$. If this is the case, then, necessarily, if $R_1<R_2$, we have that $(\mathcal{H}^{(R_2)}(z^{(R_2)},R_1),D^{(R_2)}(z^{(R_2)},R_1),z^{(R_2)})$ is isomorphic to $(\mathcal{H}^{(R_1)},D^{(R_1)},z^{(R_1)})$. Considering the direct limit with respect to R of such triples $(\mathcal{H}^{(R)},D^{(R)},z^{(R)})$, we thus obtain the existence of a graph $\mathcal{H}$ with a conductance matrix D adapted to it and with a vertex z_0 such that, for every $R>0$, $(\mathcal{H}(z_0,R),D(z_0,R),z_0)$ is isomorphic to $(\mathcal{H}^{(R)},D^{(R)},z^{(R)})$. It is a standard fact that D induces a not positively recurrent stochastic matrix Q. As a final step, it is sufficient to choose $n_k=n_k^k$ to complete the proof. ■

This allows to prove our main result:

Proof (of Theorem 3) Proposition 8 insures that Corollary 7 can be applied. The result then follows. ■

We conclude discussing an example where we do not have time-reversibility.

Example 8 (Example 6 revisited) *The sequence $P^{(n)}$ in Example 6 is complete and any limit chain is simply the biased random walk on the bi-infinite line which is known to be transient (see [15]). The reason why $\tilde{P}^{(n)}$ in Example 6 is actually non democratic follows from the fact that $\tilde{P}^{(\infty)}$ is not irreducible. Any perturbation involving a finite subset W which keeps $\tilde{P}^{(\infty)}$ irreducible, will thus be democratic because of Corollary 7.*

6 Conclusions and further problems

In this paper, we have discussed the concept of democracy for sequences of opinion dynamics and, using the language of stochastic matrices, we have given results guaranteeing the preservation of democracy under finite local perturbations. There are many issues which remain open and which, in our opinion, deserve future attention. The following is a partial list of them:

(a) It would be of interest to estimate the rate of convergence to 0 of the infinity norm of the invariant probability of the perturbed sequence considered in Theorem 6. The proof proposed does not allow for a straightforward estimation and new ideas are probably needed;
(b) What happens if the set W is grows unboundedly but remains 'small enough' with respect to V_n?
(c) Stronger variants democracy can be explored. For example we can request bounded ratios $\pi_i^{(n)}/\pi_j^{(n)}$, implying that $\pi_i^{(n)}\asymp 1/n$ for $n\to+\infty$. It would be interesting to be able to generalize our results to this stronger notion of democracy.

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